MID-SEMESTER EXAMINATION
B. MATH II YEAR, II SEMESTER 2012-2013
ANALYSIS IV.

1. Consider the set of polynomials p satisfying the condition $\int_0^1 p(x)dx = 1$ as a subset of $C[0,1]$ (with the usual supremum norm). Is this set totally bounded? Justify.

solution: Let $S = \{p(x) \ \textit{is a polinomial}\ x \in [0,1] \ \textit{such that}\ \int_0^1 p(x)dx = 1\}$ then it is easy to see that

$$S = \left\{ \sum_{k=0}^{n} a_k x^k : \sum_{k=0}^{n} \frac{a_k}{k+1} = 1, \quad 0 \leq x \leq 1, \ n \in \mathbb{N} \right\}$$

Let asssume that S is totally bounded then for each $\epsilon > 0$ there exits finite $\{p_1, p_2, \cdots, p_{m(\epsilon)}\}$ set of polinomials such that

$$S \subset \cup_{i=0}^{m(\epsilon)} B_\epsilon(p_i), \quad B_\epsilon(p_i) = \{p : \|p - p_i\|_\infty < \epsilon\}$$

Let $p_i(x) = \sum_{k=0}^{m_i} a_k^i x^k$ and define the following set

$V = \{a_k^i : i = 1, 2, \cdots, m(\epsilon) \ and\ k = 1, 2, \cdots, M\}, \ \ M = \max\{deg\, p_i : i = 1, 2, \cdots, m(\epsilon)\}.$

Since the above set V is finite we can chose $M \in \mathbb{N}$ such that

$$(M+1)\left|\frac{1}{M+1}\sum_{k=0}^{m_i} a_k^i - 1\right| > \epsilon \ \forall \ a_k^i \in V.$$

define $q(x) = (M+1)x^M$ then $q \in S$ and it is easy to see the following

$$\|q - p_i\|_\infty \geq |q(1) - p_i(1)| = (M+1)\left|\frac{1}{M+1}\sum_{k=0}^{m_i} a_k^i - 1\right| > \epsilon \ \forall \, i = 1, 2, \cdots, m(\epsilon).$$

the above is contradiction to the fact that $S \subset \cup_{i=0}^{m(\epsilon)} B_\epsilon(p_i)$, so the set S is not totally bounded. □

2. Is the set of all functions of the type $\sum_{j=0}^{N} a_j [\sin x]^{2j}$ (where $N \geq 1$ and $a_j \in \mathbb{R}$) dense in $C[-2,2]$ (with the usual supremum norm)? Justify.

solution: It is easy to see that the given set S of functions forms a subalgebra of $C[-2,2]$ and its contain identity $f(x)=1$. Let $s,t\in[-2,2]$ such that $s=-t$. Then we have $f(t)=f(-s),\ \ \forall\ f\in S$ i.e S does not separate points so we can not apply The Stone-Weierstrass Theorem.

Let $g(x)=sinx$ then $g\in C[-2,2]$ and $g(\frac{\pi}{2})=1$ and $g(-\frac{\pi}{2})=-1$. Assume S is dense in $C[-2,2]$ then there exist $\{f_n\}\in S$ such that $\|f_n-g\|_\infty\to\ 0$ as $n\to\infty$ since $f_n\in S$ we have $f_n(\frac{\pi}{2})=f_n(-\frac{\pi}{2})$. This will say that $\lim_{n\to\infty}f_n(\frac{\pi}{2})=\lim_{n\to\infty}f_n(-\frac{\pi}{2})$ but $\|f_n-g\|_\infty\to\ 0$ will imply $\lim_{n\to\infty}f_n(\frac{\pi}{2})=g(\frac{\pi}{2})=1$ and $\lim_{n\to\infty}f_n(-\frac{\pi}{2})=g(-\frac{\pi}{2})=-1$ i.e $\lim_{n\to\infty}f_n(\frac{\pi}{2})\neq\lim_{n\to\infty}f_n(-\frac{\pi}{2})$ a contradiction. So the given set of functions S is not dense in $C[-2,2]$. $\square$

3. Consider the initial value problem $y'=f(x,y),\ \ y(0)=\frac{1}{3}$, where f is a continuous function: $[-1,1]\times[-1,1]\to[-3,3]$ which has continuous partial derivative w.r.t. y at every point satisfying $|\frac{\partial f}{\partial y}|\leq 1$ at every point. Show that this problem has a unique solution on $[-\delta,\delta]$ with $\delta=\frac{2}{9}$.

solution: See existence and uniqueness theorem for IVP. $\square$

4. If f is coninuously differentiable on (a,b) and if f' is non-decreasing show that f is convex.

solution: Let $x_1<x_2<x_3$ with $x_1,x_2,x_3\in[a,b]$ then we can write

$$x_2=\frac{x_3-x_2}{x_3-x_1}x_1+\frac{x_2-x_1}{x_3-x_1}x_3.$$

Now using mean value theorem we get

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(\xi),\quad \frac{f(x_3)-f(x_2)}{x_3-x_2}=f'(\eta)\ \ x_1<\xi<x_2<\eta<x_3.$$

since f' is non-decreasing we have $f'(\xi)\leq f'(\eta)$ this will give

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\leq\frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

Re-arranging above we get

$$f(x_2)\leq\frac{x_3-x_2}{x_3-x_1}f(x_1)+\frac{x_2-x_1}{x_3-x_1}f(x_3).$$

The above imply f is convex. $\square$

5. Prove that vector space spanned by $\{z^n : n = 0, 1, 2, \cdots\}$ is not dense in the space $C(T)$ (where $T = \{z \in \mathbb{C} : |z| = 1\}$ and $C(T)$ is given supremum metric.)

solution: Let $S = \{z^n : n = 0, 1, 2, \cdots\}$ is dense in $C(T)$. Let $g(z) = \bar{z}$ then $g \in C(T)$. Since S is dense in $C(T)$ then there exist $\{f_n\} \in span(S)$ such that $\|f_n - g\|_\infty \to 0$ as $n \to \infty$. Now uniform convergence together with DCT imply

$$\int_{|z|=1} f_n(z)dz \to \int_{|z|=1} g(z)dz \quad as \quad n \to \infty.$$

Now $\int_{|z|=1} f_n(z)dz = 0$ as $\int_{|z|=1} z^k dz = 0 \;\; \forall\, k \in \mathbb{N}$ but $\int_{|z|=1} g(z)dz = 0$ a contradiction. So the vector space spanned by $\{z^n : n = 0, 1, 2, \cdots\}$ is not dense in the space $C(T)$. $\square$

6. Show that there does not exist independent elements $f_1, f_2, \cdots$ in $C[0, 1]$ which span $C[0, 1]$.

solution: Let $S = span\{f_1, f_2, \cdots\}$ then S is a vector subspace of $C[0, 1]$. w.l.o.g we assume $\|f_i\|_\infty \neq 0 \;\; \forall\, i$. Now define

$$g_n(x) = \frac{1}{2^n} \sum_{k=1}^{n} \frac{f(x)}{\|f\|_\infty} \quad n \geq 1 \quad and \quad g_n \in S.$$

Now we can see that

$$|g_n(x)| \leq \frac{n}{2^n} \quad and \quad \sum_{n=1}^{\infty} \frac{n}{2^n} < \infty.$$

so $g_n \to g$ uniformly as $n \to \infty$ therefore $g \in C[0, 1]$. It can be seen that $g_n \in S \;\; \forall\, n$ but $g \notin S$. So $S = span\{f_1, f_2, \cdots\} \neq C[0, 1]$.